2.3 Cantilever linear oscillations

Study of a cantilever oscillation is a rather science - intensive problem. In many cases the general solution to the cantilever equation of motion can not be obtained in an analytical form. However, if cantilever deflections from the equilibrium position are small, oscillations of the system will be described by well-known theories.

In chapter 2.1.1 it is shown that the Hooke's law properly describes the cantilever beam deflections from the equilibrium position. That is why small amplitude oscillations of the cantilever with one clamped end are qualified as oscillations of the spring pendulum having stiffness k and some effective mass m_{eff} [1]. The difference between the effective mass m_{eff} and the real cantilever mass is that not all cantilever oscillates with the same amplitude. The largest deflection takes place near the free end with a decay to zero at the clamped end. Chapter 2.1.6 presents calculations of the effective mass of the cantilever with given dimensions.

In this chapter we consider in detail problems of possible cantilever linear oscillations modeling it as a spring pendulum. Oscillatory systems described by linear motion equation are called linear.

2.3.1 Natural oscillations

Consider the oscillating properties of the spring pendulum, which is a point mass m, suspended from a motionless support by a massless spring having stiffness k (Fig. 1.1).

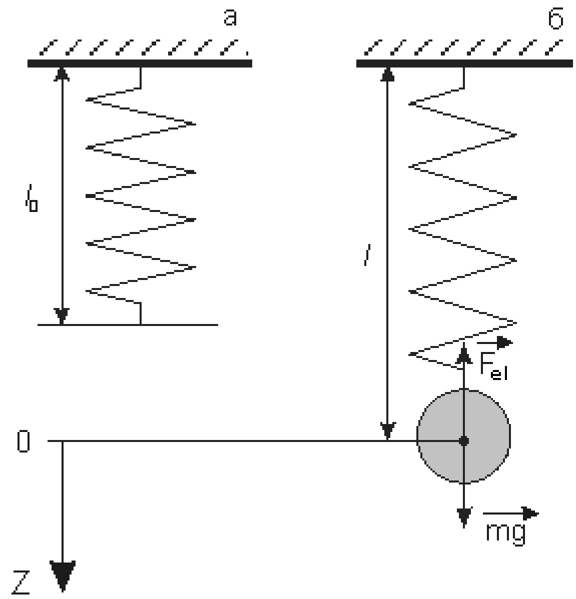


Fig.1.1. Spring pendulum.

Let l_0 be the unstretched spring length. If mass m is attached to the spring, the latter will stretch due to gravity and its length will be l. If both the mass and the spring are in equilibrium (as shown in Fig. 1.1b) the gravity is balanced by elastic force $F_{el} = -k(l - l_0)$. Let the material point coordinate be the displacement of the mass from the equilibrium position $z = 0$. Then the equation of motion can be written in the form [2 – 4]:

$$\ddot{z} + \omega_0^2 z = 0 , \tag{1.1}$$

where $\omega_0 = \sqrt{k/m}$ — frequency of natural undamped oscillations or the natural frequency. The cantilever eigenfrequency ω_0 has been calculated in chapter 2.1.6. Solution of equation (1.1) with initial conditions $z\big|_{t=0} = z_0$ and $dz/dt\big|_{t=0} = v_0$ is given by

$$z(t) = Z\cos(\omega_0 t + \varphi_0)\,, \tag{1.2}$$

$$Z = \sqrt{z_0^2 + \frac{v_0^2}{\omega_0^2}}\,,\ \varphi_0 = -\mathrm{arctg}(v_0 / z_0\omega_0)\,.$$

The amplitude and initial phase of free oscillations are determined by starting conditions for the coordinate and velocity while frequency of natural undamped oscillations is a parameter of an oscillating system.

The considered type of oscillations is usually called free oscillations because they occur in an oscillating system, which is put out of balance and left alone.

Summary:

1. Small cantilever oscillations are described by the oscillation law of the spring pendulum with given stiffness and effective mass.
2. In case of external forces absence, natural oscillations are harmonic (1.2).

2.3.2 Oscillations in the presence of friction

In chapter 2.3.1 natural oscillations of the cantilever in the absence of external forces are considered. In real systems there always takes place the dissipation of energy. If energy losses are not compensated outside, the oscillation will damp in time and eventually stop. Let us consider the spring pendulum oscillations in a viscous medium.

The frictional force acting on a body moving in a homogeneous viscous media depends only on the velocity. At small velocities the frictional force can be approximated as:

$$\mathbf{F}_{mp} = -\beta\,\mathbf{v}\,, \tag{2.1}$$

where β – constant positive factor. Taking into account the frictional force (2.1), the motion equation of the spring pendulum instead of (1.1) in chapter 2.3.1 will be written as [1-3]:

$$\ddot{z} + 2\delta\dot{z} + \omega_0^2 z = 0\,, \tag{2.2}$$

where $\delta = \beta/2m$ – damping factor.

If $\delta > \omega_0$ (case of large resistance), the solution to equation (2.2) is given by:

$$z(t) = e^{-\delta t}\left(Ae^{\alpha t} + Be^{-\alpha t}\right),\ \alpha = \sqrt{\delta^2 - \omega_0^2}\,, \tag{2.3}$$

where independent constants A and B are determined from initial conditions. As can be seen, in this case oscillations do not occur. Such motion regime is called the aperiodic.

If $\omega_0 > \delta$, the solution to equation (2.2) is written as:

$$z(t) = Z\,e^{-\delta t}\cos(\omega t + \varphi_0)\,,\ \omega = \sqrt{\omega_0^2 - \delta^2}\,, \tag{2.4}$$

$$Z = \sqrt{z_0^2 + \left(\frac{v_0 + \delta z_0}{\omega}\right)^2}\,,\ \varphi_0 = -\mathrm{arctg}\left(v_0/\omega z_0 + \delta/\omega\right)$$

In this case, the character of oscillations in the presence of the frictional force is described by a periodic function with exponentially decreasing amplitude.

In case δ and ω_0 are equal, a critical damping takes place. Equation (2.2) solution reads:

$$z(t) = e^{-\delta t}(A + Bt)\,, \tag{2.5}$$

where independent constants A and B are determined as before by initial conditions.

Fig. 2.1 shows the plot of the oscillation amplitude vs. time for different ratios between parameters δ and ω_0.

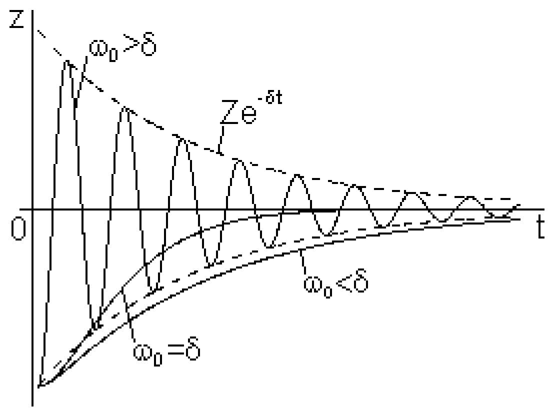


Fig.2.1. Dependence $z(t)$ for different ratios between natural frequency ω_0 and damping factor δ.

Frequently, the "quality" of an oscillatory system is characterized by dimensionless parameter Q called the quality- or Q-factor. It is proportional to the ratio between stored energy $E(t)$ and energy loss over the period $\Delta E_T = E(t) - E(t + T)$ [3]:

$$Q = \frac{2\pi E(t)}{\Delta E_T}. \tag{2.6}$$

In case of small damping ($\omega_0 >> \delta$) the total energy as a function of time is:

$$E(t) = E_0 e^{-2\delta t}, \tag{2.7}$$

where $E_0 = m\omega_0^2 Z^2/2$ — initial magnitude of the oscillator total energy. Then, in accordance with formulas (2.6) and (2.7)

$$Q \approx \frac{\pi}{\delta T} = \frac{\omega}{2\delta} \approx \frac{\omega_0}{2\delta}. \tag{2.8}$$

Thus, the Q-factor characterizes the rate of the energy transformation in a system. On the other hand, by the order of magnitude the quality factor is nothing but the number of a system oscillations over its characteristic damping time $\tau = 1/\delta$. Notice that the Q-factor not only defines the oscillations damping but is also an important quantity that determines parameters of forced oscillations under external periodic force (see chapter 2.3.3).

Summary:

1. In the presence of a frictional force, the type of natural oscillations is determined by the ratio between δ and ω_0. At $\delta > \omega_0$ aperiodic regime (3) takes place; at $\omega_0 > \delta$ oscillations are periodic with exponentially decreasing amplitude (4); at $\delta = \omega_0$ the regime of critical damping (5) exists.
2. The quality factor of an oscillating system is a very important parameter characterizing dissipative processes in a system.

2.3.3 Oscillations in the presence of external periodic driving force

Let the external periodic force act on a ball of the spring pendulum:

$$F(t) = F_0 \cos \Omega t \tag{3.1}$$

In this case the displacement of the ball from its equilibrium position is defined instead of equation (2.2) in chapter 2.3.2 by the following equation:

$$\ddot{z} + \omega_0^2 z = A_0 \cos\Omega t\,, \tag{3.2}$$

where $A_0 = F_0/m$. The solution of equation (1) at $\omega_0 \neq \Omega$ is easily written down as [1-3]:

$$z(t) = Z\cos(\omega_0 t + \varphi_0) + Z_0 \cos\Omega t \tag{3.3}$$

where $Z_0 = \dfrac{A_0}{\omega_0^2 - \Omega^2}$, $Z = \sqrt{C_1^2 + C_2^2}$, $\varphi_0 = -\mathrm{arctg}\left(-C_1/C_2\right)$, $C_1 = x_0 - \dfrac{F_0}{m(\omega_0^2 - \Omega^2)}$, $C_2 = v_0/\omega_0$. The first summand in (3.3) describes free oscillations while the second - the so called forced oscillations with amplitude A_0. Thus, if a driving force is acting, the amplitude and the initial phase of oscillations depend not only on initial conditions but on the force parameters.

In an extreme case of frequencies Ω and ω_0 coincidence, the system can not experience a periodic oscillation. The coordinate z evolution in time will be described by the following formula:

$$z(t) = Z\cos(\omega_0 t + \varphi_0) - \frac{A_0 t}{\omega_0}\sin(\omega_0 t)\,. \tag{3.4}$$

Such motion can be considered as an oscillation whose amplitude increases linearly with time. The phenomenon of oscillations amplitude unlimited growth under the periodic external force is called the resonance phenomenon.

It should be emphasized that the resonance growth of the forced oscillations amplitude beyond all bounds is an idealization of the system. Firstly, when oscillations amplitude becomes large enough, the oscillator, as a rule, is no longer the linear one. Secondly, while defining equation (3.4), the frictional forces, which damp oscillations, were not taken into consideration. Therefore, the last factor should be considered in more detail.

Forced oscillations in the presence of friction

If driving force (3.1) acts on an oscillator with friction, the equation of motion is written as:

$$\ddot{z} + 2\delta\dot{z} + \omega_0^2 z = A_0\cos\Omega t\,, \tag{3.5}$$

where δ – damping factor defined in chapter 2.2.3.2.
A general solution of (3.5) reads [1-3]:

$$z(t) = z_s(t) + Z_0\cos(\Omega t + \varphi)\,, \tag{3.6}$$

where $z_s(t)$ – equation (3.5) solution in the absence of external force (oscillator natural damped oscillations (2.3) – (2.5) in chapter 2.3.2). Due to the friction, the condition $\delta > 0$ is met and natural oscillations are damped: $z_s(t) \to 0$ at $t \to +\infty$. Therefore, over the time $t >> 1/\delta$ only forced oscillations will present in the system which are described by the second summand in (3.6). It is important to realize that parameters of forced oscillations are independent on starting conditions. Parameters of these oscillations: frequency is equal to the driving force frequency Ω, amplitude is Z_0 and phase shift is φ:

$$Z_0 = \frac{A_0}{\sqrt{(\omega_0^2 - \Omega^2)^2 + 4\delta^2\Omega^2}}\,, \tag{3.7}$$

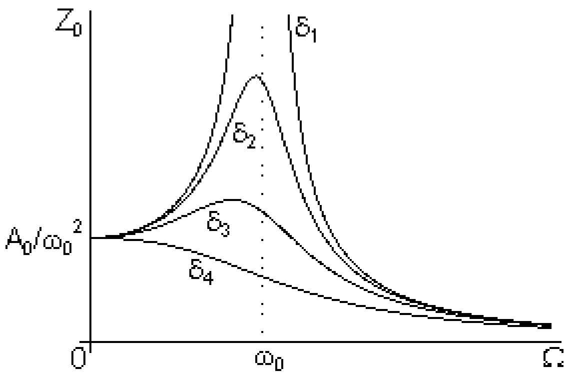
$$tg\varphi = \frac{2\delta\Omega}{\Omega^2 - \omega_0^2}\,. \tag{3.8}$$

As it arises from formula (3.8), factor δ depends on the function $\varphi(\Omega)$ derivative in the following way:

$$\frac{\partial}{\partial\Omega}\left(arctg\frac{2\delta\Omega}{\omega_0^2 - \Omega^2}\right)\Bigg|_{\Omega = \omega_0} = \delta^{-1} \tag{3.9}$$

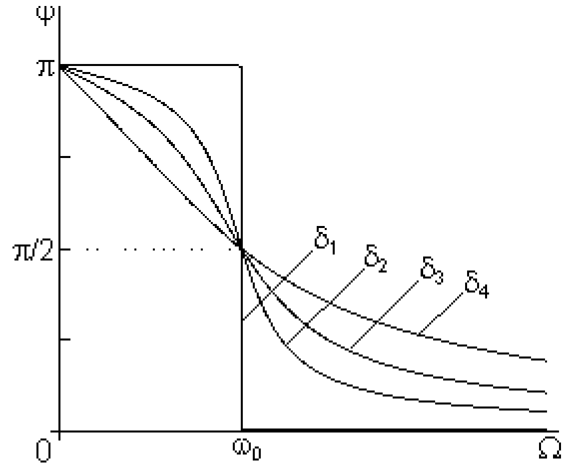
The case under study differs from the case of the forced oscillator without friction in the phase shift φ between driving force and oscillations. If the forcing frequency is equal to the natural frequency, i.e. $\Omega = \omega_0$, the phase shift is $\pi/2$ independently on the degree of damping.

Another sufficient consequence of damping is the qualitative change in the resonance curve shape. Fig. 3.1 shows relations $Z_0(\Omega)$ and $\varphi(\Omega)$ for some characteristic values of δ.



a

Fig.3.1a. Resonance curves (AFC) of the linear oscillator for different values of the friction coefficient: $\delta_1/\omega_0 = 0$, $\delta_2/\omega_0 = 0{,}2$, $\delta_3/\omega_0 = 0{,}4$, $\delta_4/\omega_0 = 0{,}8$.



б

Fig.3.1b. Graph of phase shift φ between driving force and system oscillations depending on driving frequency (PFC).

The upper bound of forced oscillations amplitude (7) is given by

$$Z_{\max} = \frac{A_0}{2\delta\sqrt{\omega_0^2 - \delta^2}} = \frac{2A_0 Q}{\omega_0^2\sqrt{4Q^2 - 1}}. \tag{3.10}$$

The corresponding resonance frequency amounts to:

$$\Omega_R = \sqrt{\omega_0^2 - 2\delta^2} = \omega_0\sqrt{1 - \frac{1}{2Q^2}}, \tag{3.11}$$

on the assumption of $\delta \le \omega_0/\sqrt{2}$. In case of light damping ($\delta \ll \omega_0$) the resonance frequency is approximately equal to the oscillator natural frequency ω_0. As δ increases, this frequency shifts to lower values (see Fig. 3.1a). At $\delta > \omega_0/\sqrt{2}$, the maximum of the forced oscillations amplitude A_0 corresponds to the frequency $\Omega = 0$. This in fact means that the resonance vanishes. Previously it was pointed out that the regime of free oscillations aperiodic damping takes place only if $\delta > \omega_0$. Hence, in the range $\omega_0/\sqrt{2} < \delta < \omega_0$, the forced oscillations are no longer of the resonant character while the oscillator motion is still oscillatory.

As it is seen from formula (3.7), for light damping δ, the forced oscillations amplitude decays rapidly as frequency goes far from the resonance one. In particular, it decreases $\sqrt{2}$ times at the following Ω values:

$$\Omega^{(\pm)} = \Omega_R\sqrt{1 \pm \frac{2\delta}{\Omega_R}\eta},\ \eta = \sqrt{\frac{\omega_0^2 - \delta^2}{\omega_0^2 - 2\delta^2}}. \tag{3.12}$$

The quantity $\Delta\Omega = \Omega^{(+)} - \Omega^{(-)}$ is called the resonant width. At small δ its magnitude is $\Delta\Omega = 2\delta$. Then the quality factor defined by formula (2.8) in chapter 2.3.2 is related to the resonant width as:

$$Q = \omega_0/\Delta\Omega. \tag{3.13}$$

Thus, the resonant width is determined by the quality factor and the natural frequency. The more is the oscillating system quality factor, the less is the resonance peak width. As follows from formula (3.13), the oscillatory system Q-factor as well as the damping factor can be estimated from experimental amplitude-frequency characteristic (AFC) of an oscillator.

Summary:

1. Oscillations of forced oscillatory system are described by periodic function, the amplitude and initial phase of oscillations being dependent not only on the starting conditions but on the driving force (3.3) parameters.
2. If forcing frequency Ω and natural frequency ω_0 are the same, the system oscillations amplitude (3.4) gradually increases, i.e. the resonance phenomenon occurs.
3. In case of forced oscillating system with friction, only forced oscillations described by the second term in (3.6) will exist in the system after time $t >> \tau$. Parameters of the forced oscillation do not depend on initial conditions. These oscillations have the frequency of driving force Ω and are characterized by amplitude (3.7) and phase shift (3.8). If frequencies coincide, i.e. $\Omega = \omega_0$, the phase shift is equal to $\pi/2$ independently on the degree of damping.
4. The resonant width is determined by the quality factor and the natural frequency (3.13).

2.3.4 Cantilever small oscillations in a force field

Consider the case when in addition to the driving force (3.1) (chapter 2.3.3), an external force $F_{ts}(z)$ acts on an oscillator. The equation of motion in this case is written as

$$\ddot{z} + 2\delta\dot{z} + \omega_0^2 z = A_0 \cos\Omega t + F_{ts}(z)/m\,. \tag{4.1}$$

Because $F_{ts}(z)$ depends only on spatial coordinates, the qualitative feature of oscillations is the same as in (3.6) (chapter 2.3.3). Force $F_{ts}(z)$ action results in the change of the oscillator equilibrium position about which the oscillations occur. For small oscillations we can take Taylor of $F_{ts}(z)$ at point z_0 corresponding to the equilibrium position:

$$F_{ts}(z) = F_{ts}(z_0) + \frac{dF_{ts}}{dz}(z_0)\,\tilde{z}(t) + o(\tilde{z}(t)^2)\,, \tag{4.2}$$

where $\tilde{z}(t)$ is expressed through $z(t)$ and z_0 as follows:

$$\tilde{z}(t) = z(t) - z_0\,, \tag{4.3}$$

and z_0 is determined from the following condition

$$\omega_0^2 z_0 = \frac{F_{ts}(z_0)}{m}\,. \tag{4.4}$$

Changing $z(t)$ in equation (4.1) by $\tilde{z}(t)$ in accordance with (4.3) and taking into account (4.4), we get

$$\ddot{\tilde{z}} + 2\delta\dot{\tilde{z}} + \tilde{\omega}^2\tilde{z} = A_0 \cos\Omega t\,, \tag{4.5}$$

where $\tilde{\omega} = \sqrt{\tilde{k}/m}$, $\tilde{k} = k - F_{ts}^{'}$, $F_{ts}^{'} = dF_{ts}/dz$, δ – damping factor determined in chapter 2.3.2.

As can be seen, equation (4.5) is identical to (3.5) in chapter 2.3.3. These equations differ in the introduction of the other spring stiffness $\tilde{k}$ and new equilibrium position z_0. Note that we can neglect the second order and higher terms in equation (4.2) only if the following condition is met

$$A_0 >> \left\langle \frac{d^2F_{ts}}{dz^2} \right\rangle \frac{Z_0^2}{m}\,, \tag{4.6}$$

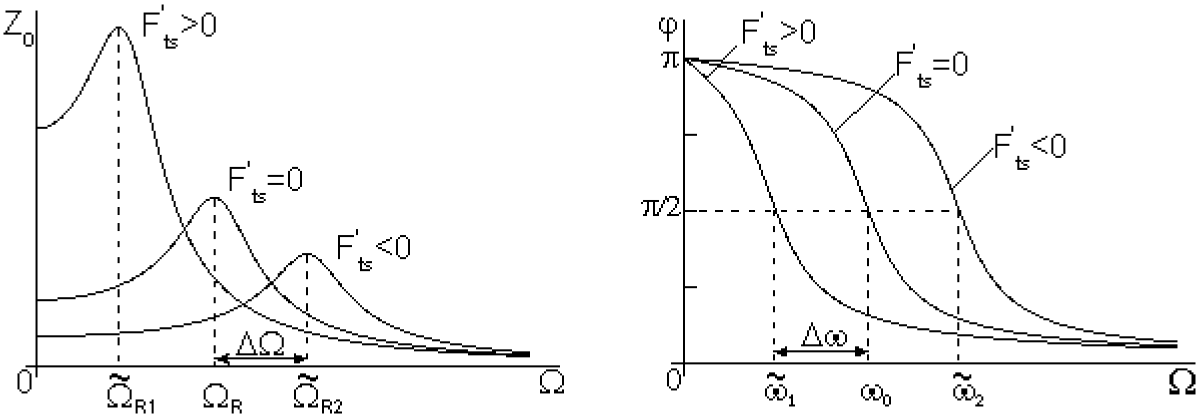
where Z_0 – oscillations amplitude at frequency ω_0 determined by formula (4.7) below. However, there are cases when $F'_{ts}=0$ and the higher order terms in (4.2) are to be taken into consideration.

Similarly to formulas (3.7), (3.8) in chapter 2.3.3 and (2.8) in chapter 2.3.2, oscillations amplitude $\tilde{Z}_0$ and phase shift $\tilde{\varphi}$ in the presence of external forces gradient is given by

$$\tilde{Z}_0=\frac{A_0}{\sqrt{\left(\tilde{\omega}^2-\Omega^2\right)^2+\frac{\omega_0^2\Omega^2}{Q^2}}}\approx\frac{\tilde{Z}_{\max}\,\tilde{\omega}/\Omega}{\sqrt{1+Q^2\left(\frac{\tilde{\omega}}{\Omega}-\frac{\Omega}{\tilde{\omega}}\right)^2}}, \qquad (4.7)$$

$$tg\varphi=\frac{\omega_0\,\Omega}{Q\left(\Omega^2-\tilde{\omega}^2\right)}, \qquad (4.8)$$

where $\tilde{Z}_{\max}=\frac{2A_0Q^2}{\tilde{\omega}^2\sqrt{4Q^2-1}}\approx\frac{A_0Q}{\tilde{\omega}^2}$ – oscillations amplitude at resonant frequency $\tilde{\Omega}_R$. Thus, the force gradient results in an additional shift of a vibrating system AFC and PFC. Fig. 4.1 shows AFC and PFC at different values of F'_{ts}.



a) b)

Fig.1. AFC – (a) and PFC – (b) at different values of F'_{ts}.

Resonant frequency $\tilde{\Omega}_R$ in the presence of external force can be written, by analogy with formula (3.10) in chapter 2.3.3, as

$$\tilde{\Omega}_R=\omega_0\sqrt{1-\frac{F'_{ts}}{k}-\frac{1}{2Q^2}}=\sqrt{\Omega_R^2-\frac{\omega_0^2F'_{ts}}{k}} \qquad (4.9)$$

Hence, the additional shift of the AFC is

$$\Delta\Omega=\tilde{\Omega}_R-\Omega_R=\Omega_R\left(\sqrt{1-\frac{\omega_0^2}{k\Omega_R^2}F'_{ts}}-1\right) \qquad (4.10)$$

If $\left|\frac{\omega_0^2}{k\Omega_R^2}F'_{ts}\right|\le 1$, we can take Taylor of the radicand in formula (4.10) to obtain:

$$\Delta\Omega\approx-\frac{\omega_0}{2k}F'_{ts} \qquad (4.11)$$

From expression (4.8) it follows that the force gradient results in the PFC shift so that its inflection point, at which the phase value is π/2, corresponds to the frequency

$$\tilde{\omega} = \omega_0\sqrt{1 - \frac{F_{ts}^{'}}{k}} \tag{4.12}$$

and

$$\Delta\omega = \tilde{\omega} - \omega_0 = \omega_0\left(\sqrt{1 - \frac{F_{ts}^{'}}{k}} - 1\right). \tag{4.13}$$

If condition $\left|\frac{F_{ts}^{'}}{k}\right| \leq 1$ is met, formula (4.13) is the same as formula (4.11).

Let us determine the phase shift $\Delta\varphi$ if there is force gradient present. If oscillations occur under the driving force at frequency ω_0, the phase shift is $\varphi = \pi/2$. In case of the force gradient presence, the phase shift in accordance with (4.8) becomes:

$$\tilde{\varphi}(\omega_0) = arctg\left(\frac{k}{QF_{ts}^{'}}\right) \tag{4.14}$$

If $\left|\frac{k}{QF_{ts}^{'}}\right| \geq 1$, we can make a Taylor's expansion of expression (4.14) as follows

$$\tilde{\varphi}(\omega_0) \approx \frac{\pi}{2} - \frac{Q}{k}F_{ts}^{'} \tag{4.15}$$

Hence, the additional phase shift due to the force gradient is (see Fig. 4.2)

$$\Delta\varphi = \tilde{\varphi}(\omega_0) - \frac{\pi}{2} \cong -\frac{Q}{k}F_{ts}^{'} \tag{4.16}$$

Consider now the amplitude change ΔA under the force gradient (see Fig. 4.3).

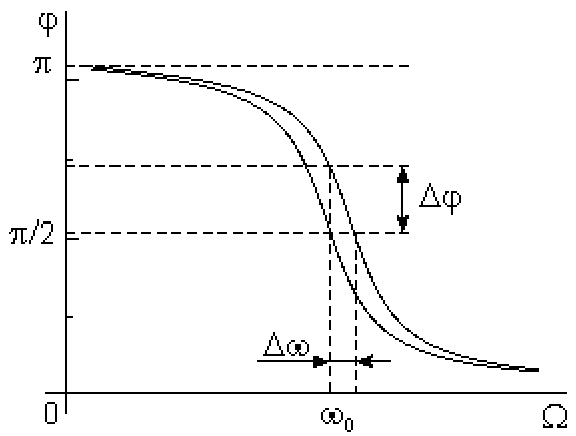


Fig. 4.2. Variation of the oscillations phase with resonant frequency.

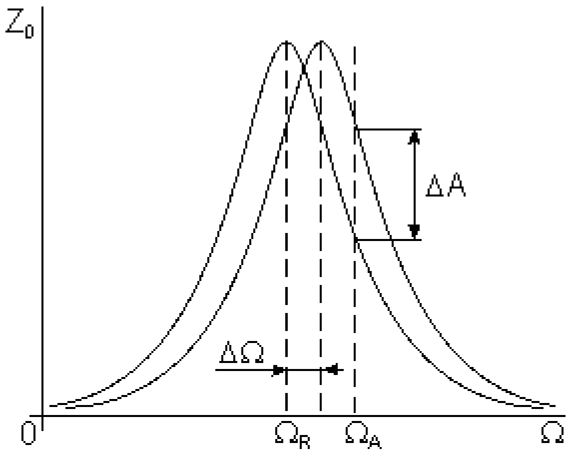


Fig. 4.3. Variation of the oscillations amplitude with resonant frequency.

The maximal change of ΔA in case of the resonant frequency (4.9) variation, takes place at certain frequencies Ω of the driving force. To these frequencies corresponds the maximum slope of the tangent to the AFC curve (linear portion of AFC):

$$\Omega_A = \omega_0\sqrt{1 - \frac{1}{k}F_{ts}^{'}}\left(1 \pm \frac{1}{\sqrt{8Q}}\right) \tag{4.17}$$

The change in oscillations amplitude (4.7) at frequency Ω_A (see Fig. 4.3) due to the force gradient in accordance with formulas (4.7) and (4.11) is given by

$$\Delta A \approx \Delta\Omega\frac{d\tilde{Z}_0}{d\Omega}(\Omega_A) \approx -\left(\frac{2\tilde{Z}_{\max}Q}{3\sqrt{3}k}\right)F_{ts}^{'} \tag{4.18}$$

The considered oscillations mode is widely used in AFM. To study the force interaction of an oscillating cantilever with a sample, in particular, one can examine the change in resonant frequency (4.11), amplitude (4.18) and phase (4.16) of oscillations and then, according to obtained data, render the force $F_{ts}(z)$ value (see, for example, chapter 2.7.1).

Summary:

1. External force $F_{ts}(z)$ action (if condition (6) is met) results only in the change of the oscillator effective stiffness (resonant frequency) and equilibrium position about which the oscillations occur. A
2. The system oscillations obey the law which is the same as in the absence of the external force.
3. The change in resonant frequency, phase and amplitude of oscillations is proportional to the external force gradient and is determined by formulas (11), (16), (18).

2.3.5 Cantilever small oscillations in a force field

Consider cantilever oscillations near a sample surface. As shown in chapter 2.2.1, the tip-sample interaction potential has a characteristic appearance depicted in Fig. 5.1. As the cantilever touches the sample and deforms its surface, the force of elastic repulsion prevails. At the tip-sample separation on the order of a few tens of angstrom, the intermolecular interaction called the Van-der-Waals force predominates.

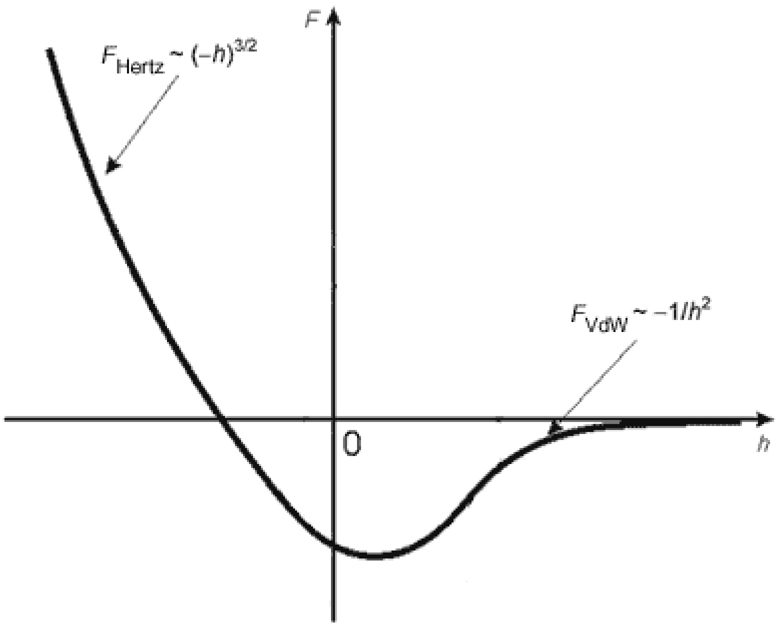


Fig.5.1. Typical appearance of the tip-sample interaction potential.

As shown in chapter 2.3.4, the presence of external force dependent on spatial coordinates, gives rise to the change in resonance properties of the cantilever-sample oscillating system.

Change in the oscillations phase $$\Delta\varphi \approx \frac{Q}{k}\frac{\partial F}{\partial z} \tag{5.1}$$

Change in the oscillations amplitude $$\Delta A \approx \left(\frac{2A_0Q}{3\sqrt{3}k}\right)\frac{\partial F}{\partial z} \tag{5.2}$$

Change in the resonant frequency $$\Delta\omega_0 \approx -\frac{1}{2k}\frac{\partial F}{\partial z}\omega_0 \tag{5.3}$$

where k – cantilever stiffness, Q – oscillating system Q-factor, A_0 – cantilever oscillations amplitude in the absence of external force.

Thus, measuring dependence of the oscillations resonant frequency, phase or amplitude on the tip-sample separation, one can render the derivative appearance and, in some cases, the interaction force itself. The corresponding experimental curves are called **the approach curves** (see Fig. 5.2).

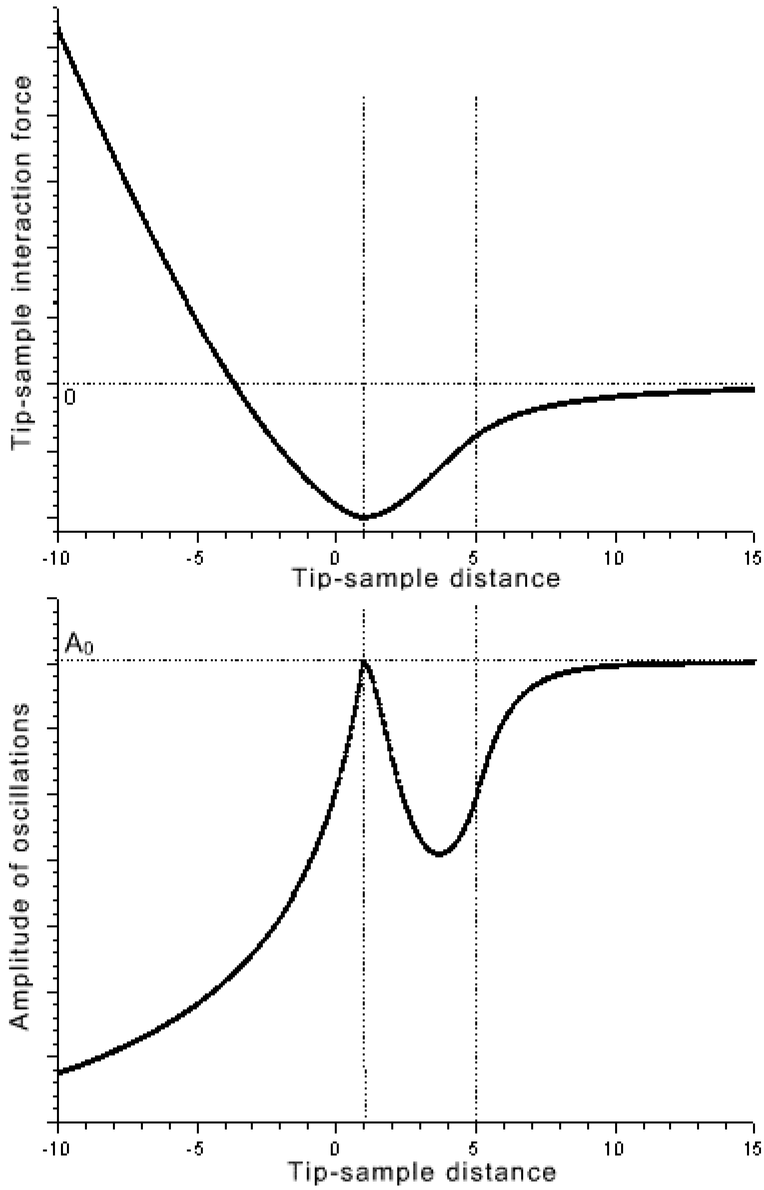


Fig.5.2. The tip-to-sample approach curves. A_0 – amplitude of the cantilever oscillations at resonant frequency, Ω_0 – resonant frequency in the absence of the external force gradient.

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