2.4 Nonlinear oscillations of cantilever

2.4.1 Cantilever nonlinear oscillations (qualitative consideration)

Consider cantilever oscillations, when in addition to driving force (see chapter 2.3.3) an external force $F_{ts}(z)$ acts on it. The equation of motion in this case is written as

$$\ddot{z} + 2\delta\dot{z} + \omega_0^2 z = A_0 \cos\Omega t + F_{ts}(z)/m \,. \qquad (1.1)$$

In a general case the steady-state solution of equation (1.1) is the sum of harmonics with frequencies divisible by a driving force frequency Ω:

$$z(t) = \sum_n A_n \cos(n\Omega + \varphi_n) \,. \qquad (1.2)$$

In chapter 2.3.4 we considered the particular case of equation (1) solution – small oscillations when the following condition is met

$$A_0 \ll \frac{m\omega_0^2}{\left\langle \frac{d^2 F_{ts}}{dz^2} \right\rangle} \qquad (1.3)$$

where ω_0 – cantilever natural resonant frequency, $\left\langle \frac{d^2 F_{ts}}{dz^2} \right\rangle$ - mean of the second derivative of the tip-sample interaction force (averaged with respect to oscillations amplitude).

In practice, condition (1.3) is seldom met. However, utilizing numerical methods, one can show that even under weak condition (1.4), the character of steady-state oscillations will only slightly differ from harmonic (a major contribution is made only by the first harmonic).

condition $$A_0 \le \frac{m\omega_0^2}{\left\langle \frac{d^2 F_{ts}}{dz^2} \right\rangle} \qquad (1.4)$$

solution $$z(t) \sim A\cos(\Omega t + \varphi) \qquad (1.5)$$

In contrast to the case of small oscillations where the steady-state condition is entirely determined by system parameters, the motion in the considered case depends on the initial state. That is, depending on the initial position of the cantilever relative to the equilibrium position, the character of the steady-state oscillations will vary (Fig. 1.1).

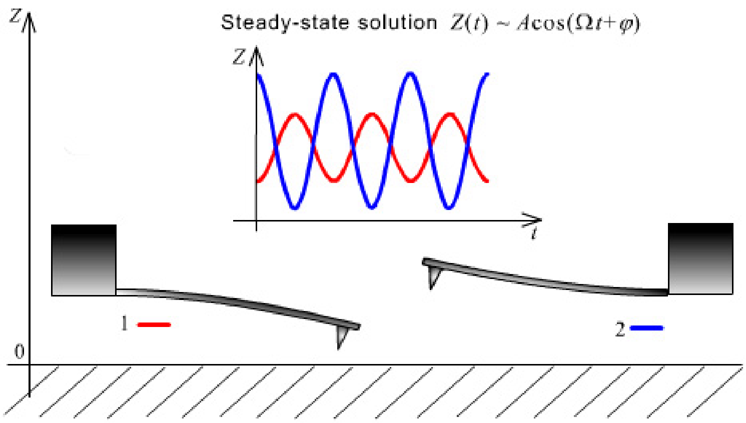

Fig. 1.1. Feasibility of several quasisteady-state solutions obtaining.

This feature results, in its turn, in the resonance curves (AFC and PFC) distortion. Fig. 1.2 shows dependences of amplitude-frequency and phase-frequency characteristics of the system on the driving force amplitude.

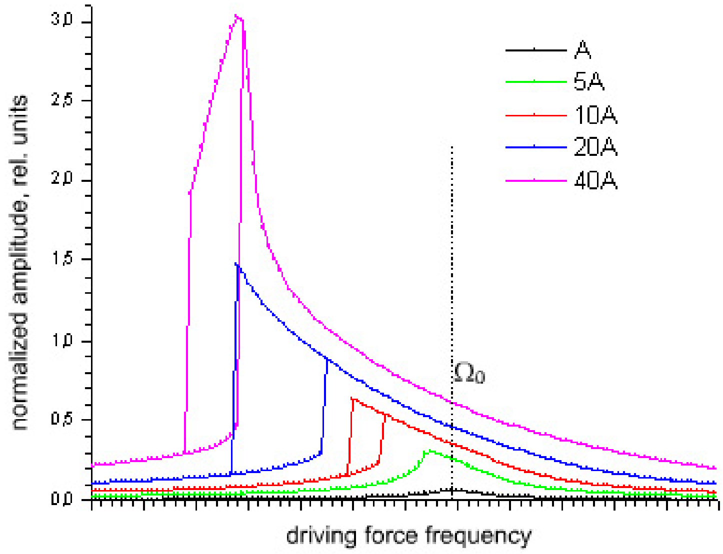

Fig. 1.2. Resonance curves in case of nonlinear oscillations.

The cantilever approach-retraction curves (see chapter 2.3.5) are distorted too, which can result in impossibility to render the character of the derivative and parameters of the tip-sample interaction force. In Fig. 1.3 are shown the cantilever approach curves for different amplitudes of the driving force.

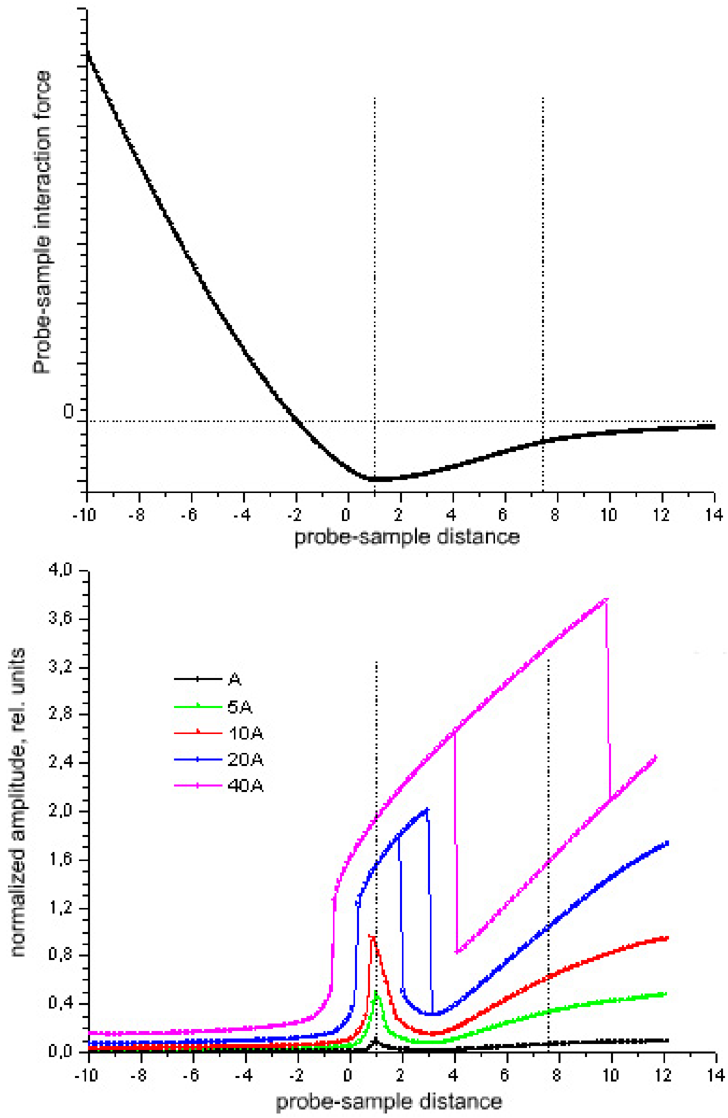

Fig. 1.3. Tip-sample approach curves for different amplitudes of the driving force.

Summary:

1. Influence of an external force $F_{ts}(z)$ in a general case is the deviation from the oscillatory motion. If, however, the condition for the driving force amplitude is weak, the oscillation is almost harmonic (a major contribution is made only by the first harmonic).
2. In contrast to the case of small oscillations where the steady-state condition is entirely determined by system parameters, the motion in the considered case depends on the initial state.

2.4.2 Analysis of cantilever motion (perturbation theory)

Consider one of the solution methods to the probe's tip equation of motion in an arbitrary potential V(z) (see formula (1.1) in chapter 2.4.1). Assume the tip of length l_{tip} is attached to the cantilever end as depicted in Fig. 2.1.

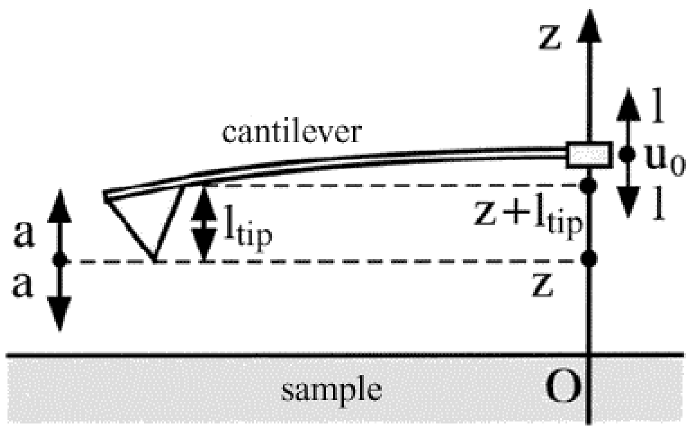

Fig. 2.1. Problem formulation.

If the free cantilever beam end is excited and oscillates with amplitude l and frequency ω at some height u_0 above a sample surface, i.e. $z_b = u_0 + l\cos(\omega t)$, the tip equation of motion is given by

$$m\frac{d^2 z}{dt^2} + \frac{m\omega_0}{Q}\frac{dz}{dt} + k(z + l_{\text{tip}} - u_0) + \frac{dV(z)}{dz} = kl\cos(\omega t), \tag{2.1}$$

where m – cantilever effective mass, Q – Q-factor, k – cantilever stiffness, ω_0 – cantilever resonant frequency, $V(z)$ – probe-sample interaction potential, z – probe-sample distance.

From condition $k\left(z_0 + l_{\text{tip}} - u_0\right) = \frac{dV(z_0)}{dz}$ one can easily determine the tip equilibrium position which is a certain function of u_0. Upon changing the variables $z = x + z_0(u_0)$ and $t = \tau/\omega_0$, the equation of motion is transformed to the following

$$m\omega_0^2 x'' + \frac{m\omega_0^2}{Q}x' + k(x + z_0(u_0) + l_{\text{tip}} - u_0) + \frac{dV\left(x + z_0(u_0)\right)}{dx} = kl\cos\left(\frac{\omega}{\omega_0}\tau\right), \tag{2.2}$$

where $x' = \dfrac{dx}{d\tau}$ and $x'' = \dfrac{d^2x}{d\tau^2}$. As the cantilever natural frequency is $\omega_0^2 = \dfrac{k}{m}$, equation (2.2) can be written as:

$$x'' + x = -\frac{1}{Q}x' - \left(z_0\left(u_0\right) + l_{\text{tip}} - u_0\right) - \frac{1}{k}\frac{dV\left(x + z_0\left(u_0\right)\right)}{dx} + l\cos\left(\frac{\omega}{\omega_0}\tau\right). \tag{2.3}$$

Introducing a new parameters designation, equation (2.3) takes the following form

$$x'' + x = \varepsilon f\left(x, x'\right) + \varepsilon A\cos\left(\Omega\tau\right), \tag{2.4}$$

$$f(x, x') = -2\mu x' - C_0 - C_1\frac{dV\left(x + z_0\left(u_0\right)\right)}{dx}, \tag{2.5}$$

$$\mu = \frac{1}{2Q\varepsilon}, \quad C_0 = \frac{z_0\left(u_0\right) + l_{\text{tip}} - u_0}{\varepsilon}, \quad C_1 = \frac{1}{\varepsilon k}, \tag{2.6}$$

$$A = \frac{l}{\varepsilon}, \quad \Omega = \frac{\omega}{\omega_0}. \tag{2.7}$$

Solution of equation (2.4) for the case $0 < \varepsilon \ll 1$ and $\Omega = 1 + O(\varepsilon)$ can be obtained with the help of Krylov-Bogolyubov-Mitropolsky (KBM) method [1] which is a kind of perturbation theory.

If the solution of (2.4) is supposed to differ slightly from that for the harmonic oscillator, i.e. $x(\tau) = a\cos\left(\Omega\tau - \Phi\right) + O(\varepsilon)$ where $a' = O(\varepsilon)$ and $\Phi' = O(\varepsilon)$ condition is met, the functions a and Φ must meet the following conditions (to within the first order terms $O(\varepsilon)$):

$$a' = -\varepsilon\mu a + \frac{\varepsilon}{2}A\sin\Phi \equiv F(a, \Phi), \tag{2.8}$$

$$\Phi' = \Omega - 1 + \varepsilon g(a, u_0) + \frac{\varepsilon}{2a}A\cos\Phi \equiv G(a, \Phi), \tag{2.9}$$

where we introduced the auxiliary function $g(a, u_0)$ determined by the probe-sample interaction potential:

$$g(a, u_0) = -\frac{1}{2\pi ka}\frac{1}{\varepsilon}\int_0^{2\pi}\frac{dV\left(z_0\left(u_0\right) + a\cos\psi\right)}{dz}\cos\psi d\psi. \tag{2.10}$$

For the definite class of equation (2.4) solutions, the cantilever end's motion is a simple harmonic motion with fixed values of a and Φ. These fixed values are the critical points of equations (2.8) and (2.9) for the case when condition $F(a, \Phi) = G(a, \Phi) = 0$ is met. Using this condition, the following equations set can be written

$$\begin{cases} -\varepsilon\mu a + \dfrac{\varepsilon}{2}A\sin\Phi = 0 \\ \Omega - 1 + \varepsilon g(a, u_0) + \dfrac{\varepsilon}{2a}A\cos\Phi = 0 \end{cases} \tag{2.11}$$

Eliminating variable Φ from the set (2.11), the oscillation amplitude dependence on the frequency, i.e. the system amplitude-frequency characteristic (AFC), can be obtained:

$$a(\Omega) = \frac{Ql}{\sqrt{1 + 4Q^2\left(\Omega - 1 + \varepsilon g(a, u_0)\right)^2}}. \tag{2.12}$$

This relation is not explicit and can not be calculated in a general form. However, using methods of numerical solution of explicit equations, the given system AFC for the specific probe-sample interaction potential can be calculated. In [2] such a calculation was performed assuming that potential $V(z)$ is the Lennard-Jones potential that is written as:

$$V(z) = \frac{\text{const}_1}{z^{12}} - \frac{\text{const}_2}{z^6} \qquad (2.13)$$

Typical system AFCs as a function of probe-sample distance are shown in Fig 2.2a. The same relations obtained by the numerical solution of the differential equation of motion (1), are depicted in Fig. 2.2b. for comparison.

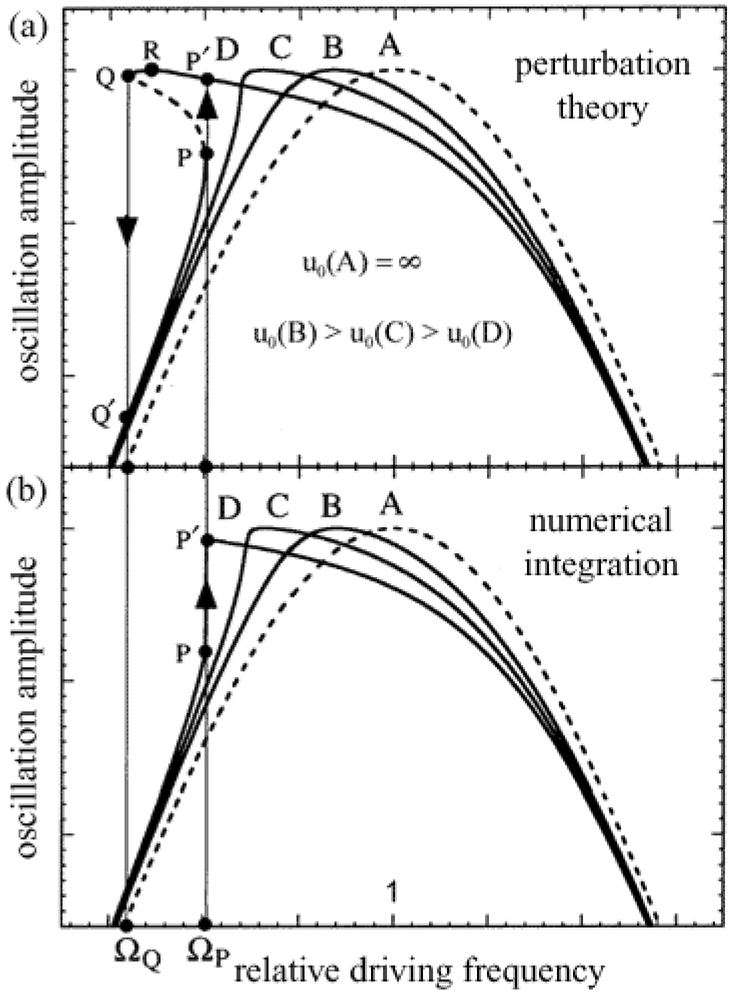

Fig. 2.2. Comparison of the tip-sample system AFC calculated using perturbation theory and numerical integration of the motion equation [2].

As is seen from the picture, when moving away from the sample surface (reduction of the interaction potential), the AFC curves approach these of the simple harmonic oscillator. The closer the tip is to the sample, the more distorted are the resonance curves. Notice that situations are available when a few stable amplitudes determined by starting conditions correspond to one driving frequency.

From (2.11) one can find the phase-frequency characteristic (PFC) of the system noticing that the following is true for Φ:

$$\text{tg}\Phi = -\frac{1}{2Q}\frac{1}{\Omega - 1 + \varepsilon g(a, u_0)}. \qquad (2.14)$$

This expression unambiguously relates the phase of oscillation to the amplitude. Thus, once system AFC is calculated, the PFC can be easily obtained using (2.14).

Summary:

1. Theoretical technique of cantilever motion equation analysis is described. This method is correct for arbitrary driving force amplitude.

2. If small oscillation condition is not satisfied (probe-sample interaction force varies very abruptly during oscillations amplitude), it is possible to exist several stable solution (state). The state choice depends on initial conditions at given driving frequency.
3. Resonance characteristics of probe-sample system are given (2.12), (2.14) at arbitrary interaction potential $V(z)$.

2.4.3 Method of resonance characteristics calculation. Approach-retraction curves

In chapter 2.4.2 we presented the approximate method of solution to the tip equation of motion in an arbitrary potential field. It was shown that the resonance characteristics of the probe-sample system are expressed by:

$$a(\Omega) = \frac{Ql}{\sqrt{1+4Q^2\left(\Omega-1+\varepsilon g(a,u_0)\right)^2}}, \tag{3.1}$$

$$\mathrm{tg}\Phi = -\frac{1}{2Q}\frac{1}{\Omega-1+\varepsilon g(a,u_0)}. \tag{3.2}$$

Equations (3.1), (3.2) relate amplitude, phase and driving force frequency implicitly. In order to facilitate the calculation of resonance characteristics consider the following method. Let us obtain from (3.1) the inverse dependence of driving force frequency on oscillations amplitude. It can be easily shown that:

$$\Omega_\pm = 1 \pm \frac{\sqrt{Q^2l^2-a^2}}{2Qa} - \varepsilon g(a,u_0). \tag{3.3}$$

Expression (3.3) describes two branches of the system AFC where «+» sign corresponds to the branch $\Omega > \Omega_R$ while «-» - to $\Omega < \Omega_R$ branch. A new designation $\Omega_R = 1-\varepsilon g(a,u_0)$ is introduced here. Now, although there are two branches instead of one, the dependence of frequency on amplitude is explicit. Next, using expression (3.2) we can obtain complete resonance characteristics of the system:

$$\Omega > \Omega_R \begin{cases} \Omega_+ = 1 + \dfrac{\sqrt{Q^2l^2-a^2}}{2Qa} - \varepsilon g(a,u_0) \\ \Phi_+ = \mathrm{atan}\left(\dfrac{a}{\sqrt{Q^2l^2-a^2}}\right) \end{cases} \tag{3.4}$$

$$\Omega < \Omega_R \begin{cases} \Omega_- = 1 - \dfrac{\sqrt{Q^2l^2-a^2}}{2Qa} - \varepsilon g(a,u_0) \\ \Phi_- = \pi - \mathrm{atan}\left(\dfrac{a}{\sqrt{Q^2l^2-a^2}}\right) \end{cases} \tag{3.5}$$

where oscillation amplitude is a parameter changing within the range $[0, Ql]$.

Both branches join at a point of maximum oscillation amplitude (resonance) $a = Ql$. To this magnitude of amplitude corresponds the driving force frequency $\Omega = \Omega_R$. Thus, Ω_R is the system resonance frequency at probe-sample distance u_0.

We can now determine how the system resonance frequency changes with the probe-sample distance. Recalling the introduced above definition of auxiliary function $g(a,u_0)$, the relative shift of the resonance frequency can be written as

$$\Delta\Omega_R = \frac{\Delta\omega}{\omega_0} = -\varepsilon g(a,u_0) = \frac{1}{2\pi ka}\int_0^{2\pi}\frac{dV\left(z_0\left(u_0\right)+a\cos\psi\right)}{dz}\cos\psi d\psi. \quad (3.6)$$

Thus, the expression for the change in the resonance frequency of the cantilever when it is retracted from the sample, contains information about the interaction potential character.

If the oscillation amplitude is small, i.e. $a \to 0$, expression (3.6) can be rewritten in the form:

$$\Delta\Omega_R = -\frac{1}{2\pi ka}\int_0^{2\pi} F_{ts}\left(z_0\left(u_0\right)+a\cos\psi\right)\cos\psi d\psi \to$$

$$\to -\frac{1}{2\pi ka}\int_0^{2\pi}\left[F_{ts}\left(z_0\left(u_0\right)\right)+\frac{dFts\left(z_0\left(u_0\right)\right)}{dz}a\cos\psi\right]\cos\psi d\psi = \quad (3.7)$$

$$= -\frac{1}{2\pi k}\frac{dF_{ts}\left(z_0\left(u_0\right)\right)}{dz}\int_0^{2\pi}\cos^2\psi d\psi = -\frac{1}{2k}\frac{dF_{ts}\left(z_0\left(u_0\right)\right)}{dz},$$

where $F_{ts}(z) = -\frac{dV(z)}{dz}$ stands for the tip-sample interaction force. Recalling the theory of a cantilever small oscillations in the field of force $F_{ts}(z)$, we can see that the expression for the resonance frequency shift is exactly the same as expression (3.7).

To obtain the amplitude-distance dependence during the tip retraction, assume that the cantilever is excited at the frequency equal to its natural resonance frequency and interaction is absent, i.e. $\omega = \omega_0$ and, accordingly, $\Omega = 1$. From the character of the system AFC it easily follows that

$$a = \frac{Ql}{\sqrt{1+4Q^2\varepsilon^2 g^2(a,u_0)}}. \quad (3.8)$$

Expression (3.8) can be written in the form $I(a,u_0) = 0$ which sets the character of the oscillation amplitude dependence on the probe-sample distance. Solution of (3.8) can in some cases be ambiguous (depending on interaction potential and probe-sample separation) that corresponds to the simultaneous existence of several oscillation modes with different amplitudes and, correspondingly, phases, because from (3.2) the phase of oscillation is the unique function of amplitude.

Consider the case of small oscillations. As it was shown for the case of the resonance frequency shift, in the given case the following is true:

$$\varepsilon g(a,u_0) = \frac{1}{2k}\frac{dF_{ts}\left(z_0\left(u_0\right)\right)}{dz}. \quad (3.9)$$

Substituting (3.9) into (3.8), we get

$$a = \frac{Ql}{\sqrt{1+\frac{Q^2}{k^2}\left(\frac{dF_{ts}\left(z_0\left(u_0\right)\right)}{dz}\right)^2}}. \quad (3.10)$$

If a vertical gradient of the tip-sample interaction force is small, the relative amplitude variation can be expressed as:

$$\frac{\Delta a}{a_0} = \frac{1}{2}\left[\frac{Q}{k}\frac{dF_{ts}\left(z_0\left(u_0\right)\right)}{dz}\right]^2. \quad (3.11)$$

Comparing the obtained expression with that derived using the theory of small amplitude oscillations, it is easily seen that both theories give exactly the same results.

Consider now the oscillation phase shift occurring during the tip retraction. Similarly to the amplitude case we assume $\Omega = 1$. Taking into account (3.2), we obtain:

$$tg\Phi = -\frac{1}{2Q}\frac{1}{\varepsilon g(a,u_0)}. \quad (3.12)$$

Using the condition for small oscillations (3.9), expression (3.12) is transformed into:

$$tg\Phi = -\frac{1}{\frac{Q}{k}\frac{dFts(z_0(u_0))}{dz}}, \quad (3.13)$$

which again corresponds exactly to that derived in the theory of small amplitude oscillations.

Summary:

1. Method of resonance characteristics calculation for probe-sample system is proposed. Using this technique frequency and phase shift at given oscillation amplitude can be found. It is more convenient than to solve the implicit system (3.1), (3.2).
2. Using equation system (3.4), (3.5) approach-retraction curves is deduced.
3. It is shown that in case of small oscillation amplitude $a \to 0$, perturbation theory coincides with theory of cantilever small oscillations.

2.4.4 Modeling of nonlinear oscillations

Let us study an oscillating cantilever behavior near a sample surface. Presented below is an interactive model that allows to investigate changes in the character of resonance and approach-retraction curves depending on the driving force amplitude and probe-sample distance.

Modeling of interaction force

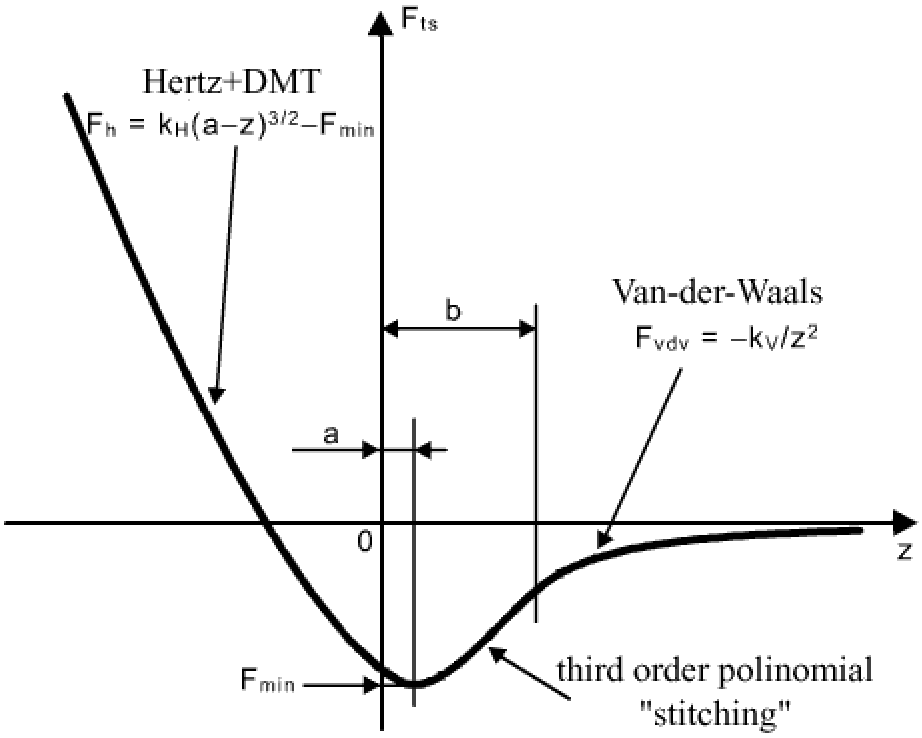

Fig. 4.1. Comparison of the tip-sample system AFC calculated using perturbation theory and numerical integration of the motion equation [2].

The interaction force is modeled as follows (Fig. 4.1):

$$F_{ts}(z) = \begin{cases} k_H(a-z)^{3/2} - F_{min} & \text{если} \quad z < a \\ C_3(z-a)^3 + C_2(z-a)^2 - F_{min} & \text{если} \quad a \le z \le b \\ -\dfrac{k_V}{z^2} & \text{если} \quad z > b \end{cases}$$

which means that just near the sample surface the force is obtained from the Herz model taking into account the adhesion in accordance with DMT model (adhesive interaction is determined by parameter F_{min}). Far enough from the surface, the Van der Waals forces dominate. Then, to join the solution, the interaction force between these regions is approximated by the third-order polynomial whose coefficients are dependent on parameters a, b, k_V, F_{min}, k_H.

Calculation of the cantilever parameters

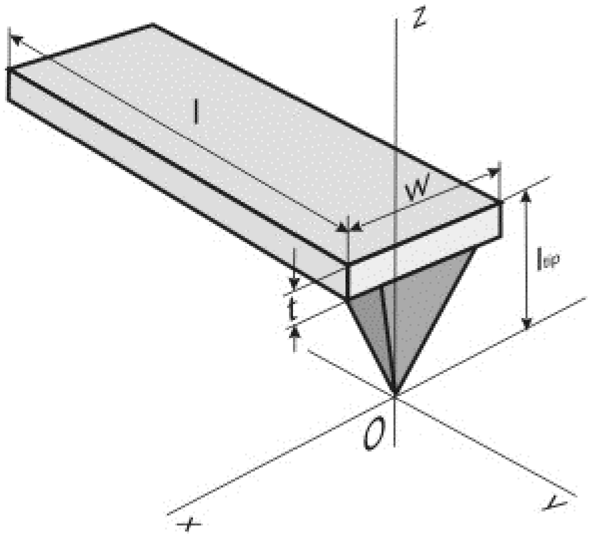

Fig. 4.2**.** Schematic geometry of cantilever**.**

It is assumed that the rectangular cantilever (Fig. 4.2) with the following sizes: l – beam length, w – width, t – thickness, l_{tip} – length of the tip, is used in this model. In accordance with the theory (see formula (12) in chapter 2.1.2) the cantilever stiffness is given by

$$k = \frac{Etw^3}{4l^3}$$

Calculation of curves

The model permits to operate in two modes: study of resonance characteristics (AFC and PFC) and study of approach-retraction curves.

Upon setting the measuring mode of resonance characteristics at a given point above a sample, sets of equations (3.4), (3.5) in chapter 2.4.3 are solved simultaneously (Fig. 4.3). The obtained curves shape depends much on the driving amplitude. To observe nonlinear effects (bistability and hysteresis) the cantilever stiffness should be minimum while system Q-factor and driving amplitude should be maximum. Also it is necessary to choose probe-sample distance short. Otherwise interaction force will be neglect and not affect on resonance characteristics.

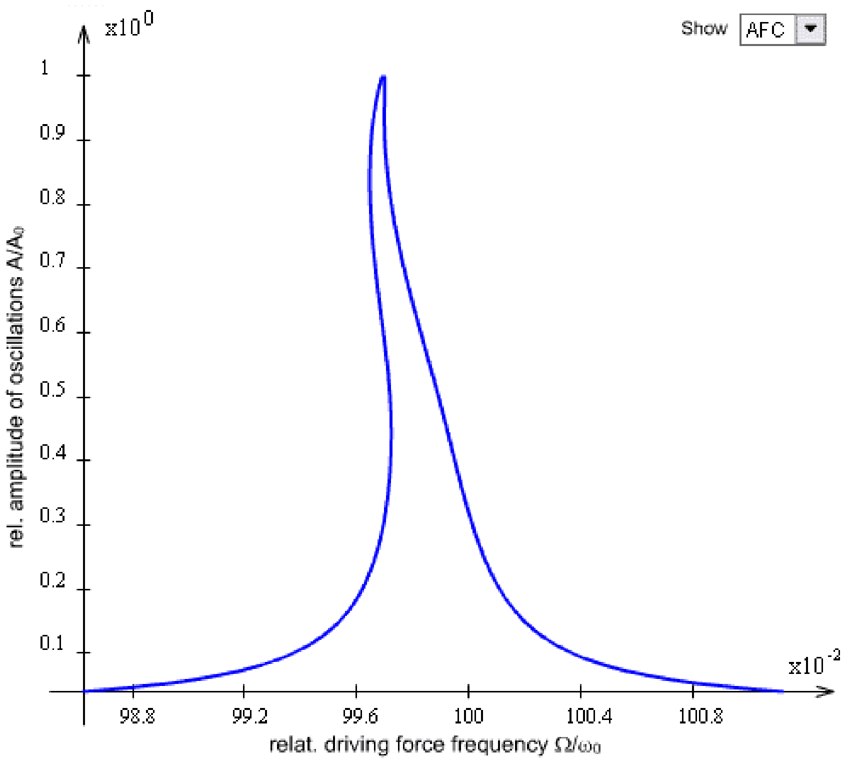

Fig. 4.3. Screen shot of the flash application. Example of the amplitude-frequency characteristic calculation.

Upon setting the measuring mode of approach-retraction curves, calculation of oscillations amplitude, resonance frequency (when maximum amplitude is reached) and phase as a function of the probe-sample distance in a given range is performed according to formulas (3.6), (3.8), (3.12) in chapter 2.4.3. When several solutions correspond to a certain tip position, small arrows show (Fig. 4.4) what curve will be measured at given scan direction (approach or retraction). Recommendation of parameters choice to obtain nonlinear effects is the same as in case of resonance characteristic.

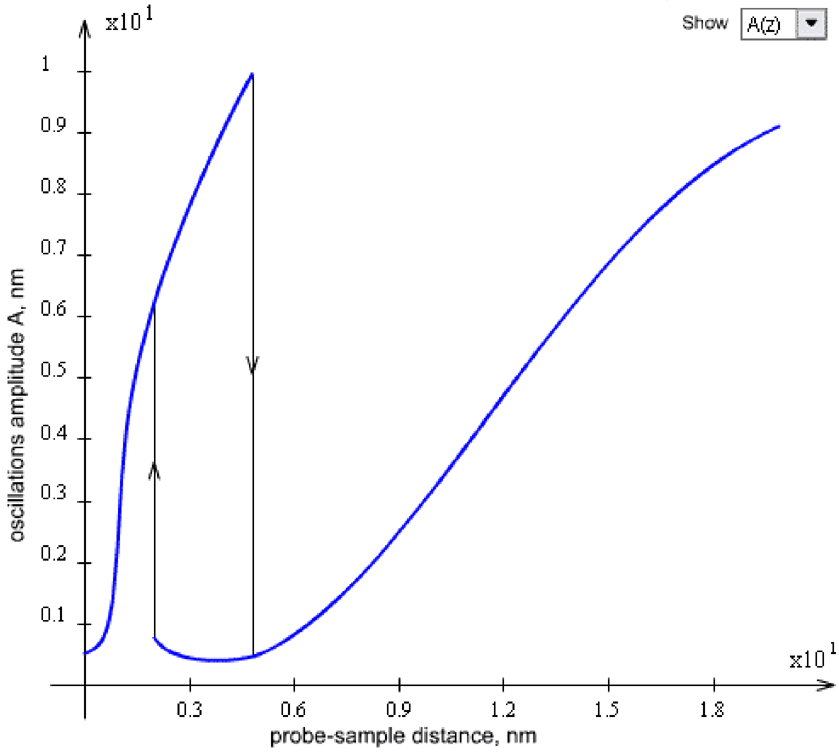

Fig. 4.4. Screen shot of the flash application. Example of the amplitude-distance curve (approach-retraction curve) calculation.

References

1. N. Bogolyubov and Y.A. Mitropolsky, Asymptotic Methods in the Theory of Nonlinear Oscillations. Gordan and Breach, New York, 1961.

2. N. Sasaki and M. Tsukada, Appl.Surf.Sci. 140 (1999) 339-343.